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Generating function for the decomposition $U(6) \supset SU(3) \times SU(2) \supset SO(3)$

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Abstract. The methods of the generating functions for invariants (covariants) of classical groups can be used for the solution of a number of problems connected with the construction of the bases of the wavefunctions and effective Hamiltonians. In particular, the method is applied to the decomposition $U(6) \supset SU(3) \times SU(2) \supset SO(3)$, and the following generating functions are obtained: (i) the generating function for the SU(3) basis of Bargmann–Moshinsky; (ii) the generating function for the number of invariants in the interacting vector boson model; (iii) the generating function for the classification of the states in the scheme $SU(3) \supset SO(3)$.

1. Introduction

Recently a number of models (microscopic and phenomenological) have been used widely for the description of the collective states of nuclei: the SU(3) model of Elliott (1958), the symplectic $Sp(6, R)$ model (Raychev 1972, Rosensteel and Rowe 1976, Fillipov *et al* 1981, Rowe 1985), the various versions of the interacting boson model (IBM) (e.g. see Arima and Iachello 1975, 1976, Jansen *et al* 1974), the interacting vector boson model (IVBM) (Georgieva *et al* 1982, 1983, 1985, 1986), the Bohr–Mottelson model (Bohr and Mottelson 1974, Faessler 1966) and so on. In all these approaches the following important problems arise:

(i) The first problem is connected with the classification of the states in these models. In the algebraic approaches the classification problem is solved by the reduction

$$G_1 \supset G_2 \supset \ldots \supset G_n \tag{1.1}$$

where G_1 is the group of dynamical symmetry and, as a rule, the last group in chain (1.1) is SO(3) (or SU(2)), whose irreducible representations (IRs) D^L (or D^J) determine the orbital (or the total) angular momentum of the nuclei L (or J). The states are labelled by the quantum numbers of the IRs of the groups in (1.1). In the general case, however, chain (1.1) is not a canonical one, i.e. in the restriction of G_i to G_{i+1} one IR of G_{i+1} may appear more than once in the IR of G_i. Thus the classification problem is reduced to the evaluation of the multiplicity ν^{Λ}_{λ} in the decomposition

$$D^{\Lambda} = \sum_{\lambda} \nu^{\Lambda}_{\lambda} D^{\lambda} \tag{1.2}$$

where D^{Λ} and D^{λ} are the IRs of the groups G_i and G_{i+1} in (1.1).

One of the most effective tools for the solution of this problem is the method of generating functions (GFs) or Molien functions (MFs) (Molien 1898). By definition, the MF $\Phi_\Lambda(x)$ is the GF for the multiplicity numbers ν_λ^Λ in (1.2):

$$\Phi_\Lambda(x)=\sum_\lambda \nu_\lambda^\Lambda x^\lambda \tag{1.3}$$

i.e. the multiplicities ν_λ^Λ in (1.2) coincide with the coefficients in the expansion of $\Phi_\Lambda(x)$ in series of the parameter x.

The IRs of the Lie groups in (1.1), referred to as symmetry groups of the model, are usually labelled by one or several integers $\lambda=(\lambda_1,\ldots,\lambda_k)$. Thus, for instance, the IRs of SO(3) are characterized by the angular momentum $L=0, 1, 2, \ldots$. For this reason these integers are chosen as exponents in (1.3) and we must consider (1.3) as a shortened form of the expression

$$\Phi_\Lambda(x_1,\ldots,x_k)=\sum_{\lambda_1,\ldots,\lambda_k} \nu^\Lambda_{\lambda_1,\ldots,\lambda_k} x_1^{\lambda_1}\ldots x_k^{\lambda_k}.$$

One can also introduce two or more parametric MFs,

$$\Phi(x,y)=\sum_{\Lambda,\lambda} \nu_\lambda^\Lambda x^\lambda y^\Lambda \tag{1.4}$$

which give the decomposition (1.2) for *all* IRs D^Λ.

The properties of GFs of the type (1.3) and (1.4) are discussed in Asherova *et al* (1988), (see also Weyl 1947). Methods for the construction of such GFs have been developed by the Montreal group (Gaskell *et al* 1978, Coutre and Sharp 1980, Giroux *et al* 1984, Gaskell and Sharp 1981, Judd *et al* 1974) (see also Gilmore and Draayer 1985).

(ii) The second problem, which can be solved by means of the MFs, is that after their reduction in the standard form, these functions give information about the structure of the basic functions $|\Lambda,\alpha,\lambda,\mathscr{K}\rangle$ of the IRs D^Λ for the reduction $G_i\supset G_{i+1}$ as a polynomial in some 'elementary permissible diagrams' (EPDs) (Moshinsky *et al* 1975). Here, the quantum number $\alpha=1,2,\ldots,\nu_\lambda^\Lambda$ distinguishes the different, linearly independent vectors, belonging to the IRs D^λ, which appear more than once in the decomposition (1.2) and $\mathscr{K}$ gives the row of the IR D^λ of the subgroup G_{i+1}. The complete set of these EPDs is, as a matter of fact, an integrity basis, which gives the structure of $|\Lambda,\alpha,\lambda,\lambda\rangle$—the highest weight vector according to G_{i+1}.

(iii) The third, very important, problem is to find the most general form of the Hamiltonian of the model. The Hamiltonian, irrespective of the particular type of interaction between the nucleons, must be represented as a function of the independent basic scalars of SO(3). We shall refer to this Hamiltonian as an effective Hamiltonian. In order to solve this problem one can use the properties of the MFs, namely, that a GF of the type

$$\Phi_0(x,\Lambda)=\sum_n \nu_n^0(\Lambda)x^n \tag{1.5}$$

gives the total number $\nu_n^0(\Lambda)$ of invariant operators with regard to the group G_i, which appear in the nth symmetrized Kronecker product $\Lambda^{[n]}$ of the IR D^Λ of the same group G_i. From the specific expression of this GF one can also establish the structure of the independent invariant operators I_k $(k=1,2,\ldots,m)$, where m is the dimension of the integrity basis of the g_i invariants (see section 2), i.e. how to construct the latter by means of the quantities $b_{\lambda\mathscr{K}}$ transforming according to the IR D^λ of G_i. The quantities $b_{\lambda\mathscr{K}}$ may be, for instance, boson creation or annihilation operators, whose quantum numbers are determined by the chain $G_i \supset G_{i+1}$. In other cases $b_{\lambda\mathscr{K}}$ may be the generators of G_i, which transform according to the adjoint representation of G_i. Then the effective Hamiltonian of the model, invariant with regards to g_i, can be expressed as a polynomial (or series) in the basic invariant operators

$$H_{\text{eff}} = \sum_{s_1,\ldots,s_m} a_{s_1,\ldots,s_m} I_1^{s_1} \ldots I_m^{s_m}. \tag{1.6}$$

Here $a_{s_1,\ldots,s_m}$ can be considered as phenomenological coefficients which can be determined, for instance, by fitting experimental data. The ordering of the operators in the RHS of (1.6) is arbitrary but fixed.

The considerations mentioned above show the importance of the MFs and their generalization for physical applications. These functions give information about the classification of the states, the structure of wavefunctions and Hamiltonians of the quantum system with a dynamical symmetry given by chain (1.1). For the sake of completeness, the definition and the general properties of the MFs will be given without proofs in section 2. Further, we will obtain some GFs, which are very useful for the construction and practical realization of IVBM. A GF for the basis of Bargmann-Moshinsky for the model SU(3) ⊃ SO(3) will be constructed in section 3. It should be noted that this basis is used in all calculations performed in IVBM.

In section 4, by means of creation and annihilation vector boson operators, we are going to construct a GF for the invariant operators in IVBM.

2. Definition and basic properties of the Molien function

The general theory of GFs can be found in Chacon *et al* (1976), Judd *et al* (1974), Gaskell *et al* (1978), Patera and Sharp (1974), Asherova *et al* (1988) and Weyl (1947).

Let b_α^λ, $\alpha = 1, 2, \ldots, [\lambda]$, be a set of operators transforming according to some reducible or irreducible representation of the (compact) group G. Here $[\lambda] = \dim \lambda$ is the dimension of the representation and α denotes the row of the representation. From b_α^λ one can construct symmetric homogeneous polynomials of degree s, which transform according to the representation $D^{[\lambda^s]}$ of G. AS a matter of fact $D^{[\lambda^s]}$ is the sth symmetrized Kronecker product of D^λ or the symmetrical plethism $[\lambda] \otimes [s]$. As a rule $D^{[\lambda^s]}$ is reducible and decomposes into a direct sum of the IRs D^Λ of the group G,

$$D^{[\lambda^s]} = \sum_\Lambda \oplus n(\Lambda, \lambda, s) D^\Lambda \tag{2.1}$$

where $n(\Lambda, \lambda, s)$ is the multiplicity of D^Λ.

The problem is to find all irreducible tensors, which are homogeneous polynomials in b_α^λ and invariant with respect to G (G-scalars) or covariants (i.e. transform according to some IR of G). It is well known (see Asherova *et al* 1988) that in both cases there exists a minimum set of G-scalars (or G-covariants), in terms of which any G-scalar (or covariant) of an arbitrary degree in b_α can be expressed in a multinomial form. This minimum set of operators is called an integrity basis for invariants (or covariants) in G.

Proposition 1. From the operators b_α^λ one can construct a set of homogeneous polynomials $I_1, \ldots, I_N, \ldots, I_{N+k}$, which are scalars with respect to G and have the following properties:

(i) the first N-invariants, called basic invariants, are algebraically independent;

(ii) an arbitrary invariant I can be represented in the form

$$I = P_0 + I_{N+1}P_1 + \ldots + I_{N+k}P_k \tag{2.2}$$

where $\{P_i\}$ are polynomials only in the basic invariants $I_1, \ldots, I_N$, $N = [\lambda]$.

The invariants $I_{N+1}, \ldots, I_{N+k}$, referred to as auxilliary invariants, appear in (2.2) at most linearly since the square of any of the auxilliary invariants can be represented as a polynomial in the basic invariants. The set of operators $I_1, \ldots, I_N, \ldots, I_{N+k}$ is referred to as an integrity basis of invariants.

Definition 1. An arbitrary homogeneous polynomial in b_α^λ, which transforms according to the IR D^Λ of G, is called a polynomial covariant P^Λ.

Proposition 2. An arbitrary covariant P^Λ can be represented in the form

$$P^\Lambda = \sum_{i=1}^{N} V_i^\Lambda P_i \tag{2.3}$$

where P_i are polynomials in the basic invariants $I_1, \ldots, I_N$ and V_i^Λ are auxilliary homogeneous polynomial covariants. The set of operators

$$I_1, I_2, \ldots, I_N, V_1, V_2, \ldots, V_N \tag{2.4}$$

is referred to as an integrity basis for covariants.

The explicit construction of the integrity basis for invariant covariants can be facilitated by making use of the MFs.

Definition 2. The MF $\Phi(\Lambda, \lambda, x)$ is a GF for the multiplicity $n(\Lambda, \lambda, s)$ in the decomposition (2.1), i.e. $n(\Lambda, \lambda, s)$ are the coefficients in the expansion of $\Phi(\Lambda, \lambda, x)$ in power series of the parameter x:

$$\Phi(\Lambda, \lambda, x) = \sum_s n(\Lambda, \lambda, s)x^2. \tag{2.5}$$

In other words $n(\Lambda, \lambda, s)$ in (2.5) give the number of the basic covariants of the type Λ and degree s, i.e. the homogeneous polynomials *in* b_α^λ *of degree* s, *which transform* according to D^Λ of G.

Theorem of Molien (1989). The GF for the number of covariants of type Λ and degree s for the finite group G is given by the expression

$$\Phi(\Lambda, \lambda, x) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_\Lambda^*(g)}{\det|E - xD^\lambda(g)|} \tag{2.6}$$

where $\chi_\Lambda(g)$ is the charactter of the element g in the IR D^Λ of G; $|G| = \dim G$ is the number of elements of the finite group G, and $D^\lambda(g)$ is the matrix of the element g in the representation D^λ.

Taking into account the properties of the characters, (2.6) can be rewritten in the form

$$\Phi(\Lambda, \lambda, x) = \frac{1}{|G|} \sum_{C \in G} n_c \frac{\chi_\Lambda^*(C)}{\det|E - xD^\lambda(C)|} \tag{2.7}$$

where n_C is the number of elements in the class C, and $D^\lambda(C)$ is the matrix of an arbitrary element $g \in C$.

Formula (2.7) can be generalized for continious groups by changing the summation over classes with integration and the order of the group $|G|$ with the group volume $V_G = \int \mathrm{d}C$.

$$\Phi(\Lambda, \lambda, x) = \frac{1}{V_G} \int \mathrm{d}C \frac{\chi_\Lambda^*(C)}{\det|E - xD^\lambda(C)|}. \tag{2.8}$$

Proposition 3. If the integrity basis of invariants consists of N basic and k auxillliary invariants of degree n_i $(1 \leqslant i \leqslant N)$ and $\mathscr{H}_j$ $(1 \leqslant j \leqslant k)$ respectively, then the MF can be reduced to the form

$$\Phi(\Lambda, \lambda, x) = \frac{1 + x^{\mathscr{H}_1} + \ldots + x^{\mathscr{H}_k}}{(1 - x^{n_1}) \ldots (1 - x^{n_N})} \tag{2.9}$$

where the numbers n_i and $\mathscr{H}_j$ may appear more than once. It should be noted that the opposite statement is not always true.

Proposition 4. If the integrity basis for covariants consists of N basic invariants and M auxilliary covariants of degrees n_i $(1 \leqslant i \leqslant N)$ and m_j $(1 \leqslant j \leqslant M)$, then the MF can be reduced to the form

$$\Phi(\Lambda, \lambda, x) = \frac{x^{m_1} + \ldots + x^{m_M}}{(1 - x^{n_1}) \ldots (1 - x^{n_N})}. \tag{2.10}$$

Again, the opposite statement is not always true.

At the end of this section we are going to give an example concerning the construction of the MF in terms of boson creation operators $b_{2\mu}^+$, which are quadrupole tensors with respect to the group SO(3). Our aim is to construct the function

$$\Phi(J, d) = \sum_{L,n} \nu(L, n) J^L d^n. \tag{2.11}$$

According to the generalization of the theorem of Molien, (2.11) can be expressed in a form analogous to (2.8), namely

$$\Phi(J, d) = \frac{1}{V_G} \int \mathrm{d}C \frac{\chi_J^*(C)}{\det|E - \mathrm{d}D^2(C)|}. \tag{2.12}$$

Here we have replaced Λ by J and have omitted the parameter $\lambda = l$, which has the value $l = 2$. In order to underline that $b_{2\mu}^+$ are SO(3) quadrupole tensors instead of $b_{2\mu}^+$ we use the letter d; in this sense the factor d^n in (2.11) reminds us that we consider a d^n configuration.

In order to obtain (2.12) in an explicit form we take into account that the class C of SO(3) is determined by a rotation through an angle $\Theta (0 \leqslant \Theta \leqslant 2\pi)$ about some axis. Then the character of the IR D^J is given by the well-known formula

$$\chi_J(\Theta) = \sum_{m=-J}^{J} \mathrm{e}^{\mathrm{i}m\Theta} = \frac{z^{J+1} - z^{-J}}{z - 1} \qquad z = \mathrm{e}^{\mathrm{i}\Theta}. \tag{2.13}$$

The volume element $\mathrm{d}C$ and the group volume V_G are given by

$$\mathrm{d}C = \sin^2 \frac{\theta}{2} \mathrm{d}\Theta \qquad V_G = \int_0^{2\pi} \sin^2 \frac{\theta}{2} \mathrm{d}\Theta = 1. \tag{2.14}$$

In the case of a rotation about the z-axis the matrix $D^2(C)$ is diagonal and has the eigenvalues

$$\mu_2 = z^2 \qquad \mu_1 = z \qquad \mu_0 = 1 \qquad \mu_{-1} = z^{-1} \qquad \mu_{-2} = z^{-2}$$

and therefore

$$\det|E - dD^2(\theta)| = (1-d)(1-dz)(1-dz^2)(1-dz^{-1})(1-dz^{-2}). \tag{2.15}$$

The integration over Θ ($z = e^{i\Theta}$) is reduced to an integration over the unit circle in the complex plain. By substituting (2.13)–(2.15) in (2.12) we obtain

$$\Phi(J, d) = -\frac{i}{\pi} \oint dz \frac{(1-z)^2}{z^2} \frac{z\chi_J^*(z)}{(1-d)(1-dz)(1-dz^2)(1-dz^{-1})(1-dz^{-2})}. \tag{2.16}$$

Then, assuming that $d < 1$ one can evaluate this integral by means of residues and after some tedious algebra we obtain

$$\Phi(J, d) = \frac{1 + d^3J^3}{(1-d^2)(1-d^3)(1-dJ^2)(1-d^2J^2)}. \tag{2.17}$$

This final expression for the MF built up of quadrupole boson creation operators coincides with formula (12) of Gaskell *et al* (1978). Its interpretation for particular cases is also well known (e.g. see Chacon *et al* 1976). However, we gave this example in order to show that, although the determination of the GFs by evaluation of integrals of the type (2.12) is straightforward, the particular calculations may be very complicated.

3. Molien function for the basis of Bargmann–Moshinsky

The basic assumption of IVBM is that the nuclear collective motions can be described by means of two types of vector bosons, called π- and ν-bosons, whose creation operators $\boldsymbol{b}_1^+$ and $\boldsymbol{b}_2^+$ are SO(3) vectors and in addition transform according to two independent IRs $(1, 0)$ of SU(3). The corresponding annihilation operators $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ transform according to $(0, 1)$ of SU(3). We also assume that π- and ν-bosons belong to a 'pseudospin' doublet and differ in an additional quantum number M_T (projection of the 'pseudospin'), which takes the value $M_T = \frac{1}{2}$ for the π-bosons and $M_T = -\frac{1}{2}$ for the ν-bosons. The corresponding 'pseudospin' operators are

$$T_1 = \frac{1}{\sqrt{2}} \boldsymbol{b}_1^+ \boldsymbol{b}_2 \qquad T_0 = \tfrac{1}{2}(\boldsymbol{b}_1^+ \boldsymbol{b}_1 - \boldsymbol{b}_2^+ \boldsymbol{b}_2) \qquad T_{-1} = \frac{1}{\sqrt{2}} \boldsymbol{b}_2^+ \boldsymbol{b}_1$$

and it is obvious that they define the Lie algebra of SU(2).

The Hamiltonian and the observables of the system must be constructed in terms of the creation and annihilation operators of these vector bosons and the basic states can be chosen as polynomials in the creation operators acting on the vacuum state. More precisely, the collective observables of such a system can be expressed in terms of the bilinear products $b_{im_i}^+ b_{jm_j}^+$, $b_{im_i} b_{jm_j}$, $b_{im_i}^+ b_{jm_j}$ ($i, j = 1, 2$; $m_i, m_j = 0, \pm 1$). These operators form the Lie algebra of the symplectic group Sp(12, R), which plays the role of the group of dynamical symmetry of the system. The set of the operators $b_{im_i}^+ b_{jm_j}$ defines the maximal compact subalgebra of Sp(12, R), namely $U(6)$.

The Hamiltonian H of the system must be an SO(3) scalar and, if we assume that it conserves the number of bosons and contains only two-body interactions, it can be expressed in terms of the U(6) generators in the following form:

$$H = \sum_{i,j} \varepsilon(i,j)\boldsymbol{b}_i^+\boldsymbol{b}_j + \sum_{i,j,k,l,L} V^L(i,j,k,l)([\boldsymbol{b}_i^+ \times \boldsymbol{b}_k]\cdot[\boldsymbol{b}_j^+ \times \boldsymbol{b}_l]^L)$$

where $\varepsilon(i,j)$ and $V^L(i,j,k,l)$ are phenomenological constants. In this case it is obvious that we can restrict ourselves and consider U(6) as the dynamical group of the system. This group has the following chains of subalgebras:

$$\begin{tikzcd}
& \mathrm{U}(6) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{O}(6) \arrow[d] \arrow[dr] & \mathrm{SU}(3)\oplus\mathrm{SU}(3) \arrow[d] & \mathrm{SU}(3)\oplus\mathrm{SU}(2) \arrow[dddl] \\
\mathrm{SU}(3)\oplus\mathrm{SO}(2) \arrow[d] & \mathrm{SO}(3)\oplus\mathrm{SO}(3) \arrow[dd] & \\
\mathrm{SO}(3)\oplus\mathrm{SO}(2) \arrow[dr] & & \\
& \mathrm{SO}(3) &
\end{tikzcd} \quad . \tag{3.1}$$

The different chains of subalgebras in (3.1) define the special symmetry limits of the model.

Hence, the problem is to find the basic SO(3) scalars, constructed from an equal number of creation and annihilation operators for the chains shown in (3.1). It is also important to build up an appropriate basis in which the Hamiltonian can be diagonalized. These problems can be solved by means of MFs.

In this paper we are interested in the rotational limit of (3.1), namely the chain U(6) ⊃ SU(3)⊕SU(2) ⊃ SO(3). In order to clarify the construction of the basis we consider a GF of the type

$$F(p,q,a,b,\xi) = \sum_{n_1,n_2,\lambda,\mu,L} \nu(n_1,u_2,\lambda,\mu,L)p^{n_1}q^{n_2}a^\lambda b^\mu \xi^L \tag{3.2a}$$

where $\nu(n_1,n_2,\lambda,\mu,L)$ is the number of SO(3) tensors of rank L, which can be constructed by n_1 creation operators of vector bosons of the type $b_{1m_1}^+$ and n_2 vector bosons of the type $b_{2m_2}^+$. By definition these tensors have a fixed (λ,μ) symmetry with regard to SU(3). Thus, this GF gives an answer to the question about the number of linearly independent states

$$|n_1,n_2,\alpha,(\lambda,\mu),\beta,L\rangle \sim (\boldsymbol{b}_1^+)^{n_1}(\boldsymbol{b}_2^+)^{n_2}|0\rangle \tag{3.2b}$$

with fixed (λ,μ) and L. Since the boson operators $b_{1m_1}^+$ and $b_{2m_2}^+$ $(m_1,m_2=0,\pm1)$ transform according to two independent IRs $(1,0)$ of SU(3), (3.2) gives an information on the following:

(i) About the reduction of the Nth symmetrized Kronecker product $N=n_1+n_2$ of the IR $(1,0)_1+(1,0)_2$ of the group SU(3) to the IR (λ,μ) of the same group,

$$[(1,0)_1+(1,0)_2]^{[N]} = \sum_{\lambda,\mu} \nu(\lambda,\mu,N)(\lambda,\mu) \tag{3.3}$$

or, which is the same, it gives the restriction of the most symmetric IR of the group SU(6) with a Young Scheme $[N]$ to the IRs of the group SU(3), where SU(6) acts in the six-dimensional space of the operators $b_{1m_1}^+$ and $b_{2m_2}^+$.

(ii) About the reduction of the IR (λ, μ) of SU(3) to SO(3),

$$(\lambda, \mu) = \sum_L \nu(\lambda, \mu, L) D^L. \tag{3.4}$$

(iii) About the explicit form of the integrity basis in terms of $b^+_{1m_1}$ and $b^+_{2m_2}$, which determines the structure of the vectors with fixed (λ, μ) and L.

In the last case the GF can be constructed by combining the GFs of the type (3.3) and (3.4).

First of all it should be noted that (3.3) can be expressed by means of the direct product of the IRs of SU(3) $(n_1, 0)$ and $(n_2, 0)$

$$(n_1, 0) \times (n_2, 0) = \sum_{z=0}^{\min(n_1, n_2)} (n_1 + n_2 - 2z, z). \tag{3.5}$$

The latter results in the following expression:

$$F_1(p_1 q, A, B) = \sum_{n_1, n_2} \sum_{z=0}^{\min(n_1, n_2)} p^{n_1} q^{n_2} A^{n_1+n_2-2z} B^z$$
$$= \frac{1}{(1-pA)(1-qA)(1-pqB)} \tag{3.6}$$

which, as a matter of fact, is a particular case of formula (11) of Patera and Sharp (1979).

The GF for (3.4) is also well known (formula (19) of Gaskell *et al* (1978)):

$$F_2(A, B, \xi) = \sum_{\lambda, \mu, L} \nu(\lambda, \mu, L) A^\lambda B^\mu \xi^L$$
$$= (1 + AB\xi) \sum_{r,s,t,u} A^{r+2t} B^{s+2u} \xi^{r+s}$$
$$= \frac{1 + AB\xi}{(1+A^2)(1-B^2)(1-A\xi)(1-B\xi)}. \tag{3.7}$$

The resulting GF $F(p_1 q, a, b, \xi)$ can be obtained by combining (3.6) and (3.7), taking into account the following considerations:

(i) In the expansion of $F_1(p, q, A, B)$ in power series,

$$F_1(p_1 q, A, B) = \sum_{n_1, n_2, \lambda, \mu} \nu_1(n_1, n_2, \lambda, \mu) p^{n_1} q^{n_2} A^\lambda B^\mu$$

$\nu_1(n_1, n_2, \lambda, \mu)$ gives the multiplicity of the IR (λ, μ) of SU(3) in the decomposition of the Kronecker product $[(n_1) \times (n_2)]^N$.

(ii) In the expansion of $F_2(A, B, \xi)$ in power series,

$$F_2(A, B, \xi) = \sum_{\lambda', \mu', L} \nu_2(\lambda', \mu', L) A^{\lambda'} B^{\mu'} \xi^L$$

$\nu_2(\lambda', \mu', L)$ gives the multiplicity of the IR D^L of SO(3) in the IR (λ', μ') of SU(3).

(iii) The multiplicity $\nu(n_1, n_2, \lambda, \mu, L)$ in (3.2) is obviously given by

$$\nu(n_1, n_2, \lambda, \mu, L) = \nu_1(n_1, n_2, \lambda, \mu) \nu_2(\lambda, \mu, L).$$

Now let us consider the product function

$$F_1 F_2 = \sum_{n_1, n_2, \lambda, \mu, \lambda', \mu', L} \nu_1(n_1, n_2, \lambda, \mu) \nu_2(\lambda', \mu', L) p^{n_1} q^{n_2} A^{\lambda+\lambda'} B^{\mu+\mu'} \xi^L.$$

If we take into account only terms with $\lambda = \lambda'$ and $\mu = \mu'$ we obtain

$$F_1 F_2 = \sum_{n_1, n_2, \lambda, \mu, L} \nu_1(n_1, n_2, \lambda, \mu) \nu_2(\lambda, \mu, L) p^{n_1} q^{n_2} A^{2\lambda} B^{2\mu} \xi^L.$$

Further, if $\sqrt{a}$ and $\sqrt{b}$ are substituted for A and B this expansion can be represented in the form (3.2):

$$F_1(p_1q, \sqrt{a}, \sqrt{b}) F_2(\sqrt{a}, \sqrt{b}, \xi) = \sum_{n_1, n_2, \lambda, \mu, L} \nu(n_1, n_2, \lambda, \mu, L) p^{n_1} q^{n_2} a^{\lambda} b^{\mu} \xi^{L}.$$

Taking into account (3.6) and (3.7) the resulting GF can be expressed in the form

$$\begin{aligned} F(p, q, a, b, \xi) &\\ = \sum_{x, y_1 z} & p^{x+z} q^{y+z} (\sqrt{a})^{x+y} (\sqrt{b})^{z} \\ \times \sum_{r_1 s_1 t, u} & [(\sqrt{a})^{r+2t} (\sqrt{b})^{s+2u} \xi^{r+s} + (\sqrt{a})^{r+2t+1} (\sqrt{b})^{s+2u+1} \xi^{r+s+1}]. \end{aligned} \tag{3.8}$$

The condition that only terms with equal powers of the parameters A and B in F_1 and f_2 should be taken into account leads to the restrictions $x+y=r+2t$, $z=s+2u$ or $x+y=r+2t+1$, $z=s+2u+1$ and after the summation we obtain

$$F(p_1q, a, b, \xi) = \frac{1 + a^2pq + abp^2q\xi + abpq^2\xi - a^3pq^2\xi - a^3p^2q\xi - a^2bp^2q^2\xi^2 - a^4b^3q^3\xi^2}{(1-pqb\xi)(1-p^2q^2b^2)(1-aq\xi)(1-a^2q^2)(1-ap\xi)(1-a^2p^2)}. \tag{3.9}$$

This function has the following important properties:

(i) It is symmetric with regard to the substitution $\boldsymbol{b}_1^+ \leftrightarrow \boldsymbol{b}_2^+$, i.e. $p \leftrightarrow q$.

(ii) If $a=b=1$ the GF (3.2) reduces to

$$\begin{aligned} F(p_1q, 1, 1, \xi) &= \sum_{n_1, n_2, \lambda, \mu, L} \nu(n_1, n_2, \lambda, \mu, L) p^{n_1} q^{n_2} \xi^{L} \\ &= \sum_{n_1, n_2, L} \bar{\nu}(n_1, n_2, L) p^{n_1} q^{n_2} \xi^{L} \end{aligned}$$

where

$$\bar{\nu}(n_1, n_2, L) = \sum_{\lambda, \mu} \nu(n_1, n_2, \lambda, \mu, L)$$

is the multiplicity of L-tensors that appear in the symmetric Kronecker product $[l=1 \oplus l=1]^{[N]}$ for arbitrary (λ, μ) of SU(3). In this case from (3.9) one obtains

$$F(p_1q, \xi) = \frac{1+pq\xi}{(1-p^2)(1-p\xi)(1-q^2)(1-q\xi)(1-pq)}. \tag{3.10}$$

This formula coincides with formula (14) of Gaskell *et al* (1978). It gives the classification of the states (3.2*b*) $|n_1, n_2, \gamma, L, M\rangle$ without taking account of the SU(3) symmetry, that is, γ includes the labels α, (λ, μ) and β in (3.2*b*).

(iii) If $a=b=1$ and $\xi=0$ the GF (3.9) reduces to the MF for SO(3) invariant operators, which can be constructed by means of two types of vector bosons. In this case (3.9) can be rewritten in the form

$$F_0(p_1q) = \frac{1+pq}{(1-p^2q^2)(1-q^2)(1-p^2)} = \frac{1}{(1-pq)(1-q^2)(1-p^2)} \tag{3.11}$$

which is a particular case of (3.10) and is in accordance with similar results (Gilmore and Draayer 1985).

(iv) If $q=0$, $a=1$ formula (3.9) reduces to the GF for the multiplicity of SO(3) tensors, constructed by vector bosons of only one type, namely, the product $[l=1]^{[N]}$.

In this case (3.9) has the form

$$F(p_1\xi)=\frac{1}{(1-p\xi)(1-p^2)} \tag{3.12}$$

which is in agreement with Asherova *et al* (1988).

The classification of the states for $n_1+n_2=N\leqslant 4$ following from (3.9) is given in table 1. One can easily verify that (3.12) is also in accordance with table 1. Thus the GF (3.9) gives a complete classification of the states in IVBM with fixed n_1, n_2, L and (λ, μ).

It should be pointed out, however, that (3.9) is not yet the GF for the basis of Bargmann and Moshinsky. In their original work (Bargmann and Moshinsky 1961), the authors assume that b_1^+ and b_2^+ are united in an SU(2) doublet (a 'pseudospin' doublet) and differ in the third projection of the 'pseudospin' $\tau=\pm\frac{1}{2}$. Further, they consider only vectors of the type (3.2*b*) with a fixed total 'pseudospin' $T=\lambda/2$ and maximal value of the 'pseudospin' projection

$$M_T=\tfrac{1}{2}(n_1-n_2)=T=\frac{\lambda}{2}. \tag{3.13}$$

In this case the GF for the basis can be constructed by means of calculations analogical to (3.9) under the additional restriction (3.13). As a final result we obtain

$$F(p_1q, a, b, \xi)=\frac{1+p^2qab\xi}{(1-pa\xi)(1-pqb\xi)(1-p^2a^2)(1-p^2q^2b^2)}. \tag{3.14}$$

Table 1. Classification of the states in IVBM ($N=n_1+n_2=4$).

n_1	n_2	(λ, μ)	L
0	0	(0, 0)	0
1	0	(1, 0)	1
0	1	(1, 0)	1
2	0	(2, 0)	0, 2
1	1	(2, 0)	0, 2
		(0, 1)	1
0	2	(2, 0)	0, 2
3	0	(3, 0)	1, 3
2	1	(3, 0)	1, 3
		(1, 1)	1, 2
1	2	(3, 0)	1, 3
		(1, 1)	1, 2
0	3	(3, 0)	1, 3
4	0	(4, 0)	0, 2, 4
3	1	(4, 0)	0, 2, 4
		(2, 1)	1, 2, 3
2	2	(4, 0)	0, 2, 4
		(2, 1)	1, 2, 3
		(0, 2)	0, 2
1	3	(4, 0)	0, 2, 4
		(2, 1)	1, 2, 3
0	4	(4, 0)	0, 2, 4

According to propositions 3 and 4 (see (2.9 and 2.10)) this GF has the following meaning: each term of the type $p^{n_1}q^{n_2}a^{\lambda}b^{\mu}\xi^{L}$ in the denominator of (3.14) corresponds to the basic covariants, i.e. SO(3) tensors of rank L, which are constructed by n_1 π-bosons and n_2 ν-bosons and transform according to the IR (λ, μ) of SU(3). The same is valid for the term in the numerator, but this covariant is an auxiliary one, i.e. it can appear at most linearly.

Thus the basis of Bargmann and Moshinsky (1961),

$$|N=n_1+n_2,(\lambda,\mu),T=\frac{\lambda}{2},M_T=\tfrac{1}{2}(n_1-n_2)=\frac{\lambda}{2},L,M=L\rangle$$

is constructed as a stretched product of the EPDs, given in table 2 and can be represented in the following form:

$$\left|\begin{matrix} N=\lambda+2\mu,\ T=M_T=\dfrac{\lambda}{2} \\ \alpha,L,L\end{matrix}\right\rangle_{\mathrm{BM}}=w^{\beta}\eta_1^{a}A_1^{b}(\eta^2)^{c}A_{12}^{d}|0\rangle$$

The basic EPDs of the integrity basis η_1, η^2, A_1 and A_{12} can appear in arbitrary degrees, while w appears at most linearly. The latter is due to the fact that $w^2=\eta^2A_1-\eta_1^2A_{12}$. The integers a, b, c, d and β are determined by the conditions

$$L=\beta+a+b$$

$$N=\lambda+2\mu=3\beta+a+2b+2c+4d$$

$$T=\frac{\lambda}{2}=\tfrac{1}{2}(\beta+a+2c).$$

It is obvious that

$$\lambda+\mu=\beta+L+2c+2d$$

which leads to

$$\beta=\begin{cases}0 & \text{if } \lambda+\mu-L \text{ even}\\ 1 & \text{if } \lambda+\mu-L \text{ odd}\end{cases}$$

and finally for the basis of Bargmann and Moshinsky we have

$$\left|\begin{matrix}(\lambda,\mu)\\ \alpha,L,L\end{matrix}\right\rangle_{\mathrm{BM}}=w^{\beta}\eta_1^{L+\mu+2\alpha}A_1^{\mu-2\alpha-\beta}(\eta^2)^{(1/2)(\lambda+\mu-1-2\alpha-\beta)}A_{12}^{\alpha}|0\rangle \tag{3.15}$$

$$\max\{0,\tfrac{1}{2}(\mu-L)\}\leqslant\alpha\leqslant\min\{\tfrac{1}{2}(\mu-\beta),\tfrac{1}{2}(\lambda+\mu-L-\beta)\}.$$

Table 2. EPDs for the construction of the basis of Bargmann–Moshinsky.

Term in (3.14)	n_1	n_2	λ	μ	L	EPDs in (3.15)
$pa\xi$	1	0	1	0	1	$\eta_1=(b_1^+)_1$
p^2a^2	2	0	2	0	0	$\eta^2=(b_1^+\cdot b_1^+)$
$pqb\xi$	1	1	0	1	1	$A_1=[b_1^+\times b_2^+]_1^1$
$p^2q^2b^2$	2	2	0	2	0	$A_{12}=([b_1^+\times b_2^+]^1\cdot[b_1^+\times b_2^+]^1)$
$p^2qab\xi$	2	1	1	1	1	$w=[b_1^+\times[b_1^+\times b_2^+]^1]_1^1$

4. Generating function for the invariant operators of IVBM

The GF for the L-tensors, which can be constructed by the vector creation operators $\boldsymbol{b}_1^+$ and $\boldsymbol{b}_2^+$ is of the type (3.10), If $p=q$, (3.10) can be rewritten in the form

$$F_1(p_1\xi)=\frac{1+p^2\xi}{(1-p^2)^3(1-p\xi)^2}. \tag{4.1}$$

By analogy to (4.1) the GF for the L-tensors built from the annihilation operators $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ is

$$F_2(\bar{p},\xi)=\frac{1+\bar{p}^2\xi}{(1-\bar{p}^2)^3(1-\bar{p}\xi)^2}. \tag{4.2}$$

Our purpose is to construct a GF for the SO(3) invariants, which can be constructed by $\boldsymbol{b}_1^+$, $\boldsymbol{b}_2^+$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ under the additional condition of conservation of the boson number. The latter means that only terms of zero degree in ξ and equal powers of the parameters p and $\bar{p}$ should be taken into account in the product $F_1(p_1\xi)f_2(\bar{p},\xi^{-1})$. Further, the product $p^N\bar{p}^N$ must be substituted by s^N and as a result one obtains the GF

$$\mathscr{F}(s)=\sum_N \nu(N)s^N \tag{4.3}$$

where $\nu(N)$ is the number of invariants of the type

$$(\boldsymbol{b}_1^+)^a(\boldsymbol{b}_2^+)^b(\boldsymbol{b}_1)^c(\boldsymbol{b}_2)^d$$

with $a+b=c+d=N$.

Taking into account (4.1) and (4.2) we start from the expression

$$\mathscr{F}(s)=(1+p^2\xi)(1+\bar{p}^2\xi^{-1})\sum p^{2a+2b+2c+d+e}\bar{p}^{2a'+2b'+2c'+d'+e'}\xi^{d+e-d'-e'} \tag{4.4}$$

which splits into four terms:

(i) The first term is of the type

$$S_1=\sum p^{2a+2b+2c+d+e}\bar{p}^{2a'+2b'+2c'+d'+e'}\xi^{d+e-d'-e'}$$

where the summation is carried out under the conditions

$$a+b+c=a'+b'+c'$$

$$d+e=d'+e'.$$

The result for S_1 is

$$S_1=(1+s)(1+4s^2+s^4)[(1-s^2)^5(1-s)^3]^{-1}.$$

(ii) The term S_4 gives the same result multiplied by s^2.

(iii) The terms S_2 and S_3 are of the type

$$S_2=\sum p^{2a+2b+2c+2+d+e}\bar{p}^{2a'+2b'+2c'+d'+e'}\xi^{d+e+1-d'-e'}$$

under the conditions

$$d+e+1=d'+e'$$

$$2a+2b+2c+1=2a'+2b'+2c'$$

which is impossible for a, b, c, d, e, a', b', c', d', e' integers. Thus S_2 and S_3 are equal to zero and the GF is

$$\mathscr{F}(s)=(1+s^2)S_1=(1+s+5s^2+5s^3+5s^4+5s^5+s^6+s^7)[(1-s)^3(1-s^2)^5]^{-1}. \tag{4.5}$$

The latter means that the integrity basis for the effective Hamiltonian of IVBM, which conserves the boson number, consists of:

(i) three basic invariants of first degree with respect to the creation and annihilation operators;

(ii) five basic invariants of second degree;

(iii) five auxiliary invariants of second, third, fourth and fifth degrees respectively;

(iv) one auxiliary invariant of first, one of sixth and one of seventh degree.

Further, it is of interest to discuss the second-degree invariant operators, because these operators give the potential energy of the interacting bosons. Expanding (4.5) in a Taylor series one obtains

$$\mathscr{F}(s)=1+4s+19s^2+56s^3+\ldots.$$

The second-degree invariants can be constructed by combining the operators

$$J_1^\lambda=[b_1^+\times b_1^+]^\lambda\lambda=0,2 \qquad J_2^\lambda=[b_2^+\times b_2^+]\lambda=0,2 \qquad J_3^\lambda=[b_1^+\times b_2^+]^\lambda\lambda=0,1,2$$

with the analogous operators $\bar{J}_i^\lambda$ built up from the annihilation operators. The tensors with $\lambda=0$ give rise to nine scalar combinations of the type $(J_i^0\bar{J}_k^0)i$, $k=1$, 2, 3; with $\lambda=2$, to another set of nine invariants, and the last invaraint is $(J_3^1\bar{J}_3^1)$. Here we have not taken into account the hermicity of the operators J_i^λ, $\bar{J}_k^\lambda$, which will reduce the number of the independent invariants.

As mentioned above, (4.5) give the GF for the SO(3) invariants that conserve the total number of the vector bosons $N=n_1+n_2$, but do not conserve the total 'pseudospin' T and its third projection. That is why it will be very useful to construct the SO(3) invariants under the additional conservation of either T or M_T.

In the case of T-conservation the GF for the SO(3) invariants can be constructed by means of two GFs of the general type (3.14) with $p=q=1$:

$$\begin{aligned}\mathscr{F}_1(a,b,L)&=\frac{1+abL}{(1-aL)(1-bL)(1-a^2)(1-b^2)}\\ \mathscr{F}_2(\bar{a},\bar{b},\bar{L})&=\frac{1+\bar{a}\bar{b}\bar{L}}{(1-\bar{a}\bar{L})(1-\bar{b}\bar{L})(1-\bar{a}^2)(1-\bar{b}^2)}.\end{aligned} \tag{4.6}$$

For the decomposition $SU(3)\supset SO(3)$ the power of the parameter a is equal to λ, the power of b is equal to μ, and the boson number N and the 'pseudospin' T are given by $N=\lambda+2\mu$ and $T=\lambda/2$. The latter means that the conservation of both N and T can be ensured by keeping only terms with equal powers of the parameters a, $\bar{a}$ and b, $\bar{b}$ in the product $\mathscr{F}_1\mathscr{F}_2$ and then substituting $a^{n_1}\bar{a}^{n_1}$ and $b^{n_2}\bar{b}^{n_2}$ by $s_1^{n_1}$ and $s_2^{n_2}$ respectively. The SO(3) invariance is ensured by taking L and $\bar{L}$ in the same powers. Then the GF can be expressed as

$$\mathscr{F}(s_1,s_2,\xi)=\sum_{n_1,n_2,k}\nu(n_1,n_2,k)s_1^{n_1}s_2^{n_2}\xi^k \qquad \xi=L\bar{L}$$

where $\nu(n_1,n_2,k)$ is the number of invariants of the type

$$(b_1^+)^a(b_2^+)^b(b_1)^c(b_2)^d$$

with $a+b=c+d=N$ and $T=0$. This function can be calculated by starting from the expression

$$\mathscr{F}(s_1, s_2, \xi) = (1+s_1 s_2 \xi) \sum a^{2k+c} \bar{a}^{2k'+c'} b^{2e+d} \bar{b}^{2e'+d'} L^{c+d} \bar{L}^{c'+d'} \tag{4.7}$$

under the conditions

$$\begin{aligned} 2k+c &= 2k'+c' \\ 2e+d &= 2e'+d' \\ c+d &= c'+d' \end{aligned} \tag{4.8}$$

where the first two conditions follow from the conservation of N and T, and the third one from the SO(3) invariance. Finally, one obtains

$$\mathscr{F}(s_1, s_2, \xi) = (1+s_1^2 s_2^2 \xi^2)[(1-s_1\xi)(1-s_2\xi)(1-s_1^2)(1-s_2^2)(1-s_1 s_2 \xi)]^{-1}. \tag{4.9}$$

As mentioned above, the powers of the parameters a and b in (4.7) are identified with the SU(3) labels λ and μ respectively and, since $N = \lambda + 2\mu$, it is evident that the parameter $s_2 = b\bar{b}$ symbolizes an invariant of second order with respect to the invariant associated with the parameter $s_1 = a\bar{a}$. At this point it is reasonable to transform (4.9) into a one-parametrical form with a parameter s controlling the total degree of the invariant. It can be done by means of the substitutions $s_1 = s$, $s_2 = s^2$ and then (4.9) is

$$\mathscr{F}(s, \xi) = \frac{1+s^6\xi^2}{(1-s\xi)(1-s^2\xi)(1-s^2)(1-s^4)(1-s^3\xi)}. \tag{4.10}$$

Following the same line of reasoning as in the case of (4.5) it can be concluded that the integrity basis for the effective IVBM Hamiltonian, which conserves N, L and T, is realized by means of:

(i) one basic invariant of first order—the term $s\xi$;
(ii) two basic invariants of second order—the terms s^2 and $s^2\xi$;
(iii) one basic invariant of third order—the term $s^3\xi$;
(iv) one basic invariant of fourth order—the term s^4;
(v) one auxiliary invariant of sixth order—the term $s^6\xi^2$ in the numerator of (4.10).

Again, as in the case of (4.5), in the effective Hamiltonian the basic invariants can appear in arbitrary degrees, while the auxiliary one can appear at most linearly. On the other hand, following the results of previous investigations (Raychev and Roussev 1981) it is well known that the Hamiltonian with a dynamical symmetry SU(3) ⊃ SO(3) can be expressed as

$$H = H_0 + V$$

where H_0 is invariant with regard to SU(3) and V decreases the SU(3) symmetry to SO(3). The operator V, being an SO(3) scalar, can be expressed as a polynomial in the SO(3) basic scalars in the enveloping algebra of SU(3) (Judd *et al* 1974), i.e.

$$V = V(C_2, C_3, L^2, X^{(3)}, X^{(4)}, X^{(6)}). \tag{4.11}$$

Here C_2 and C_3 are the second- and third-order Casimir operators of SU(3). The operator L^2 is of the form

$$L^2 = (L \cdot L)$$

$X^{(3)}$ and $X^{(4)}$ are given by

$$\begin{aligned} X^{(3)} &= ([L \times Q]^1 \cdot L) \\ X^{(4)} &= ([L \times Q]^1 \cdot [L \times Q]^1) \end{aligned} \tag{4.12}$$

and $X^{(6)}$ is expressed by the commutator

$$X^{(6)}=[X^{(3)},X^{(4)}]$$

where L_μ and Q_μ are the components of the angular and SU(3) quadrupole operators respectively. In the operator V (4.11) the invariants C_2, C_3, L^2, $X^{(3)}$ and $X^{(4)}$ play a basic role and appear in arbitrary degrees, while the invariant $X^{(6)}$ is an auxiliary one and appears at most linearly.

It has also been shown (Afanas'ev *et al* 1972, Raychev and Roussev 1981) that there exists another operator, which is an SO(3) scalar and splits the SU(3) multiplets. This operator is very convenient for calculations on the basis of Bargmann–Moshinsky and is of the following type:

$$\Omega=A^+A \tag{4.13}$$

where A^+ coincides with the EPD A_{12} in (3.16) and can also be expressed by

$$A^+=(b_1^+)^2(b_2^+)^2-(b_1^+b_2^+)^2$$

and A is hermitian conjugate to A^+. According to a theorem proved in (Judd *et al* 1974) the operator Ω can be expanded as a polynomial in the basic SO(3) scalars in (4.11) and this expansion will give the connection between Ω and $X^{(4)}$. For particular physical problems, however, one can use either Ω or $X^{(4)}$. The matrix elements of the operators $X^{(3)}$ and Ω are calculated in Raychev and Roussev (1981).

Now, having in mind the tensorial structure of the invariants in the GF (4.10) and V (4.11), it can be shown that:

(i) the basic invariant $s\xi$ is identified with $N=b_1^+b_1+b_2^+b_2$;
(ii) the basic invariants s^2 and $s^2\xi$ are identified with C_2 and L^2;
(iii) the basic invariant $s^3\xi$ is identified with $X^{(3)}$ (4.12);
(iv) the basic invariant s^4 is identified with $X^{(4)}$ (4.12) or Ω (4.13);
(v) the auxiliary invariant $s^6\xi^2$ is identified with the commutator

$$[X^{(3)},X^{(4)}] \qquad \text{or} \qquad [X^{(3)},\Omega].$$

It should be mentioned that the dynamical symmetry of the IVBM in the rotational limit is given by the group chain

$$\mathrm{U}(6)\supset\mathrm{U}(3)\times\mathrm{U}(2)\supset\mathrm{SO}(3) \tag{4.14}$$

where, because of the boson realization of the generators, the direct product representation of U(3) × U(2) is embedded in the most symmetric representation of U(6). In this case the U(3) and U(2) labels are given by

$$[N]_6=\sum_{i=0}^{[N/2]}[N-i,i,0]_3[N-i,i]_2 \qquad \left[\frac{N}{2}\right]=\begin{cases}\dfrac{N}{2} & N \text{ even}\\[2mm] \dfrac{N-1}{2} & N \text{ odd}\end{cases} \tag{4.15}$$

i.e. only U(3) representations of the type $[\lambda,\lambda_2,0]$ are possible. It is well known (Vanagas 1971) that the Casimir operators C_1, $C_2,\ldots,C_r,\ldots$ of the group U(n) are independent only for U(n) representations with $\lambda_r\neq0$. If $\lambda_r=0$ and $\lambda_{r-1}\neq0$ the operator C_r can be expressed as a polynomial in C_1, $C_2,\ldots,C_{r-1}$. Hence, in the case (4.15) the operator C_3 is not independent and can be expresed as a polynomial in C_1 and C_2, which explains the fact that in the structure of the GF (4.10) appears only one basic invariant of third order identified with the operator $X^{(3)}$.

It should also be pointed out that the groups U(3) and U(2) in (4.14) are complementary (Moshinsky and Quesne 1971) in the sense that the eigenvalues of the second-order Casimir operator of U(3) are uniquely determined by the eigenvalues of the second-order Casimir operator of U(2). This is due in the relation

$$C_2(\mathrm{U}(3)) = \tfrac{3}{2}C_2(\mathrm{U}(2)) + N \tag{4.16}$$

which means that the IRs of U(3) and U(2) can be labelled by the same quantum numbers, for instance the boson number N and the 'pseuodospin' T. Following this line of reasoning it is evident that according to (4.16) one of the second-order invariants in (4.10) can be identified with T^2, where T^2 is connected with $C_2(\mathrm{U}(2))$ by

$$C_2(U(2)) = \tfrac{4}{3}T^2 + \tfrac{1}{3}N^2.$$

In this way we proved that the integrity basis for the IVBM Hamiltonian, which conserves the boson number N, and the 'pseudospin' T consists of five basic invariants N, L^2, T^2, $X^{(3)}$, Ω and one auxiliary invariant $X^{(6)} = [X^{(3)}, \Omega]$.

As mentioned above, the conservation of the 'pseudospin' is not obligatory. Thus, for instance, the consideration of the relation between IVBM and IBM-2 would lead to the separation of the $\pi\bar{\pi}$ and $\nu\bar{\nu}$ pairs from the $\pi\bar{\nu}$ and $\nu\bar{\pi}$ pairs, which means that it is the third projection of the 'pseudospin' $M_T = \frac{1}{2}(n_1 - n_2)$, but not T, that must be conserved. In this case the GF can be constructed by means of two, two-parametrical GFs of the type (3.10):

$$F_1(p, q, L) = (1 + pqL)[(1-p^2)(1-q^2)(1-pq)(1-pL)(1-qL)]^{-1}$$

$$F_2(\bar{p}, \bar{q}, L^{-1}) = (1 + \bar{p}\bar{q}L^{-1})[(1-\bar{p}^2)(1-\bar{q}^2)(1-\bar{p}\bar{q})(1-\bar{p}L^{-1})(1-\bar{q}L^{-1})]^{-1}.$$

Further, again in the product $F_1(p_1q, L)F_2(\bar{p}, \bar{q}, L^{-1})$ only terms with zero power with respect to L and equal powers of the parameters p and $\bar{p}$, and q and $\bar{q}$, should be taken into account. Then the products $p^{n_1}\bar{p}^{n_1}$ and $q^{n_2}\bar{q}^{n_2}$ must be substituted by s^{n_1} and s^{n_2} respectively. The result gives the GF for the SO(3) invariants that conserve N and M_T. This function is of the following type:

$$\mathscr{F}(s) = \frac{1 + s^2 + 2s^3 + 4s^4 + 2s^5 + s^6 + s^8}{(1-s)^2(1-s^2)^3(1-s^3)^2}. \tag{4.17}$$

5. Conclusions

In a forthcoming paper we are going to discuss in more detail the relation between IVBM and IBM-2, where the function (4.17) will play an important role.

Also, it should be mentioned that some of the results of this paper can find an additional region of application. In fact the GF (3.10) corresponds to the decomposition $\mathrm{U}(6) \supset \mathrm{SO}(3)$ and gives the multiplicity of the different IRs D^L of SO(3) in the Nth symmetric product of the six-dimensional IR $[1]_6$ of $U(6)$, which decomposes in IRs of SO(3) according to the rule $[1]_6 = D^1 + D^1$. From the point of view of SU(3) we assumed that

$$[1]_6 = (1, 0) + (1, 0) = D^1 + D^1. \tag{5.1}$$

If instead of (5.1) one uses

$$[1]_6 = (1, 0) + (0, 1) = D^1 + D^1 \tag{5.2}$$

then (3.5) transforms into

$$(n_1, 0) \times (0, n_2) = \sum_{z=0}^{\min(n_1, n_2)} (n_1 - z, n_2 - z)$$

and the GF of the type (3.6) will be

$$\begin{aligned} F(p_1 q, A, B) &= \sum_{n_1, n_2} \sum_{z=0}^{\min(n_1, n_2)} p^{n_1} q^{n_2} A^{n_1 - z} B^{n_2 - 2} \\ &= [(1 - pA)(1 - qB)(1 - pq)]^{-1}. \end{aligned} \tag{5.3}$$

The combination of (5.3) and (3.7), taking into account only terms of zero degree with regard to the parameters A and B, again results in the GF (3.10). Hence, the GF (3.10) is applicable to both cases (5.1) and (5.2). The latter explains the similarity between (3.10) and (3.7), namely that (3.10) can be obtained from (3.7) by substituting A with p and B with q, and then multiplying the result by the factor $(1 - pq)^{-1}$, i.e.

$$F(p, q, \xi) = \frac{F_2(p_1 q, \xi)}{1 - pq}.$$

The SU(3) substructure (5.2) appears in the translationally invariant three-body problem, where the Jacobi coordinates are given by

$$\boldsymbol{t}_1 = \frac{1}{\sqrt{2}} (\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad \boldsymbol{t}_2 = \frac{1}{\sqrt{6}} (\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3)$$

(the masses are equal). Further, one can introduce creation and annihilation operators of oscillator quanta $\boldsymbol{b}_1^+$, $\boldsymbol{b}_2^+$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ and transform them to the operators

$$\boldsymbol{b}_\pm^+ = \frac{1}{\sqrt{2}} (\boldsymbol{b}_1^+ \pm \mathrm{i} \boldsymbol{b}_2^+) \qquad \boldsymbol{b}_\pm = \frac{1}{\sqrt{2}} (\boldsymbol{b}_1 \pm \mathrm{i} \boldsymbol{b}_2).$$

The operators $\boldsymbol{b}_+^+$ and $\boldsymbol{b}_-^+$ transform according to $(1, 0)$ and $(0, 1)$ of SU(3) respectively. The SU(3) algebra is defined by

$$A_{i,k} = b_i^+ b_k - b_{-k}^+ b_{-i} \qquad i, k = x, y_1 z.$$

In this way the set of operators $\boldsymbol{b}_+^+$, $\boldsymbol{b}_-^+$ determines a basis for the six-dimensional IR $[1]_6$ of U(6) with an SU(3) substructure given by (5.2). Thus (3.10) gives the GF not only for IVBM but for the three-body problem as well in the case of the decomposition $U(6) \supset SO(3)$.

References

Afanas'ev G N, Avramov S R and Raychev P P 1972 *Sov. J. Nucl. Phys.* **16** 53
Arima A and Iachello F 1975 *Phys. Rev. Lett.* **35** 1069
—— 1976 *Ann. Phys.* **99** 253
Asherova R M, Zhilinsky B I, Pavlov-Verevkin V B and Smirnov Yu F 1988 ITP-88-70P Kiev
Bargmann V and Moshinsky M 1961 *Nucl. Phys.* **23** 177
Bohr A and Mottelson B 1974 Nuclear Structure (New York: Benjamin)
Chacon E, Moshinsky M and Sharp R T 1976 *J. Math. Phys.* **17** 668
Coutre E and Sharp R T 1980 *J. Phys. A: Math. Gen.* **13** 1925
Elliott J P 1958 *Proc. R. Soc.* A **245** 128
Faessler A 1966 *Nucl. Phys.* **85** 653

Fillippov G F, Ovcharenko V I and Smirnov Yu F 1981 *Microscopic Theory of Nuclear Collective Excitations* (Kiev: Naukova Dumka)
Gaskell R, Peccia A and Sharp R T 1978 *J. Math. Phys.* **19** 727
Gaskell R and Sharp R T 1981 *J. Math. Phys.* **22** 2756
Georgieva A, Raychev P and Roussev R 1982 *J. Phys. G: Nucl. Phys.* **8** 1377; 1983 *J. Phys. G: Nucl. Phys.* **9** 521; 1985 *Bulg. J. Phys.* **12** 2
Georgieva A, Raychev P and Roussev R 1986 *Proc. Int. Conf. Nuclear Structure, Reactions and Symmetries (Dubrovnik)* vol 13 (Chur: Harwood) p 97
Gilmore R and Draayer J. P 1985 *J. Math. Phys.* **26** 3053
Giroux Y, Coutre M and Sharp R T 1984 *J. Phys. A: Math. Gen.* 17 715
Jansen D, Jolos R and Donau F 1974 *Nucl. Phys.* A **224** 93
Judd B R, Miller W, Patera J and Winternitz P 1974 *J. Math. Phys.* **15** 1787
Molien T 1898 *Berliner Sitzungsberichte* (Berlin: Berein) p 1152
Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1791
Moshinsky M, Patera J, Sharp R T and Winternitz P 1975 *Ann. Phys.* **95** 139
Patera J and Sharp R T 1974 *Lecture Notes Phys.* **94** 175
Raychev P P 1972 *Sov. J. Nucl. Phys.* **16** 643
Raychev P and Roussev R 1981 *J. Phys. G: Nucl. Phys.* **7** 1227
Rosensteel G and Rowe D J 1976 *Ann. Phys.* **96** 1
Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419
Vanagas V V 1971 *Algebraic Methods in Nuclear Theory* (Vilnius: Mintis)
Weyl H 1947 *Classical Groups* (Moscow: Nauka)